Stat 201: Introduction to Statistics

Significance Tests – for Mean Differences

Telling Which Parameter We're After

- As statisticians, or data scientists, it's our job to hear a problem and decide what we're after
  - We call the parameter of interest the **target parameter**

Parameter	Point Estimate	Key Phrase	Type of Data
$\mu_1 - \mu_2$	$\overline{x_1} - \overline{x_2}$	Mean Difference	Quantitative
$\rho_1 - \rho_2$	$\widehat{p_1} - \widehat{p_2}$	Difference of Proportion, percentage, fraction, rate	Qualitative (Categorical)

Note!

- We use the usual approach of confidence intervals and hypothesis testing on the mean difference $\mu_d = \mu_1 - \mu_2$ as we did in chapters 7-9
  - Our data becomes $x_{d_i} = x_{1_i} - x_{2_i}$ and we are interested in making inference on μ_d.

Hypothesis Test for Mean Difference: Step 1

- State Hypotheses:
  - **Null hypothesis**: that the population mean equals some μ_o
    - $H_o: \mu_d = \mu_1 - \mu_2 \leq \mu_o$ (one sided test)
    - $H_o: \mu_d = \mu_1 - \mu_2 \geq \mu_o$ (one sided test)
    - $H_o: \mu_d = \mu_1 - \mu_2 = \mu_o$ (two sided test)
  - **Alternative hypothesis:** What we're interested in
    - $H_a: \mu_1 - \mu_2 > \mu_o$ (one sided test)
    - $H_a: \mu_1 - \mu_2 < \mu_o$ (one sided test)
    - $Ha: \mu_1 - \mu_2 \neq \mu_o$ (two sided test)

Hypothesis Test for Mean Difference: Step 2

- Check the assumptions
  - The difference variable must be quantitative
  - The data are obtained using randomization
  - Samples are **independent**
  - The differences are from the normal distribution
    - If $n_1 > 30 \; and \; n_2 > 30$
    - If either n<30 we need to check a histogram to ensure the differences are normally distributed

Hypothesis Test for Mean Difference: Step 3: Case One With known σ_1 & σ_2

- Calculate Test Statistic
  - The test statistic measures how different the sample proportion we have is from the null hypothesis
  - We calculate the z-statistic by assuming that $\mu_{d\,0}$ is the population mean difference

$$z^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Hypothesis Test for Mean Differences: Step 4: Case One With known σ_1 & σ_2

- Determine the P-value
  - The P-value describes how unusual the sample data would be if H_o were true.

Alternative Hypothesis	Probability	Formula for the P-value
$H_a: \mu_1 - \mu_2 > \mu_o$	Right tail	P(Z>z*)
$H_a: \mu_1 - \mu_2 < \mu_o$	Left tail	P(Z<z*)
$H_a: \mu_1 - \mu_2 \neq \mu_o$	Two-tail	2*P(Z<-|z*|)

Hypothesis Test for Mean Difference: Step 3: Case Two With unknown $\sigma_1 = \sigma_2$

- Calculate Test Statistic
  - The test statistic measures how different the sample proportion we have is from the null hypothesis
  - We calculate the t-statistic by assuming that μ_{d_0} is the population mean difference

$t^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where $s_p^2 = \frac{(\mathrm{n}_1 - 1)\mathrm{s}_1^2 + (\mathrm{n}_2 - 1)\mathrm{s}_2^2}{n_1 + n_2 - 2}$

Hypothesis Test for Mean Difference: Step 4: Case Two With unknown $\sigma_1 = \sigma_2$

- Determine the P-value
  - The P-value describes how unusual the sample data would be if H_o were true.

Alternative Hypothesis	Probability	Formula for the P-value
$H_a: \mu_1 - \mu_2 > \mu_o$	Right tail	P(T>t*)
$H_a: \mu_1 - \mu_2 < \mu_o$	Left tail	P(T<t*)
$H_a: \mu_1 - \mu_2 \neq \mu_o$	Two-tail	2*P(T<-|t*|)

Hypothesis Test for Mean Difference: Step 3: Case Three With unknown $\sigma_1 \neq \sigma_2$

- Calculate Test Statistic
  - The test statistic measures how different the sample proportion we have is from the null hypothesis
  - We calculate the t-statistic by assuming that μ_{d_0} is the population mean difference

$$t^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Hypothesis Test for Mean Difference: Step 4: Case Three With unknown $\sigma_1 \neq \sigma_2$

- Determine the P-value
  - The P-value describes how unusual the sample data would be if H_o were true.

Alternative Hypothesis	Probability	Formula for the P-value
$H_a: \mu_1 - \mu_2 > \mu_o$	Right tail	P(T>t*)
$H_a: \mu_1 - \mu_2 < \mu_o$	Left tail	P(T<t*)
$H_a: \mu_1 - \mu_2 \neq \mu_o$	Two-tail	2*P(T<-|t*|)

Hypothesis Test for Mean Difference: Step 5

- Summarize the test by reporting and interpreting the P-value
  - Smaller p-values give stronger evidence against H_o
- If p-value$\leq (1 - confidence) = \alpha$
  - Reject H_o, with a p-value = _____, we have sufficient evidence that the alternative hypothesis might be true
- If p-value$> (1 - confidence) = \alpha$
  - Fail to reject H_o, with a p-value = _____, we do not have sufficient evidence that the alternative hypothesis might be true

Example

- According to a NY Times article a survey conducted showed that 22 men averaged 3 hours of housework per day with a standard deviation of .85 and 49 women averaged 6 hours of housework per day with a standard deviation of 1.3
- Test with 90% confidence interval that the true population difference of means is more males than females

Example

- State Hypotheses:
  - $H_o: \mu_d = \mu_1 - \mu_2 \leq 0$
  - $H_a: \mu_d = \mu_1 - \mu_2 > 0$

Example

- Check the assumptions
  1. Each sample must be obtained through randomization
  2. Samples are **independent**
  3. The differences are from the normal distribution
     - If $n_1 < 30$ & $n_2 > 30$

       AND
     - We don't know the populations follow the normal distribution

     Proceed with caution

Example

- Calculate Test Statistic

$$t^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$= \frac{((3 - 6) - 0)}{\sqrt{\frac{.85^2}{22} + \frac{1.3^2}{49}}}$$

$$= -11.56151$$

Example

First we solve for v:

$$v = \frac{\left(\frac{.85^2}{22} + \frac{1.3^2}{49}\right)^2}{\frac{\left(\frac{.85^2}{22}\right)^2}{22-1} + \frac{\left(\frac{1.3^2}{49}\right)^2}{49-1}} = 59.5402 \approx 59$$

Example

- Determine the P-value
  - The P-value describes how unusual the sample data would be if H_o were true.

$$1 - P(T < t^*)$$
$$= 1 - P(T < -11.56151\ )$$
$$\approx 1 - 0 = 1$$

Example

- Summarize the test by reporting and interpreting the P-value

$1 > (1 - .9) = .1$

We fail to reject H_o, we do not have sufficient evidence to suggest that males do more housework than females

Summary!

Sampling Distribution for the Sample Mean Summary

Shape, Center and Spread of Population	Shape of sample	Center of sample	Spread of sample
Populations are normal with means μ and standard deviations σ.	Regardless of the sample size n, the shape of the distribution of the sample mean is normal	$\mu_d = \mu_1 - \mu_2$	$\sigma_d = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
Population are not normal with means μ and standard deviations σ.	As the sample size n increases, the distribution of the sample mean becomes approximately normal	$\mu_d = \mu_1 - \mu_2$	$s_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Hypothesis Testing for μ known σ_1 and σ_2

Step One:	(i) $H_0: \mu_1 - \mu_2 = \mu_0$ & $H_a: \mu_1 - \mu_2 \neq \mu_0$ (ii) $H_0: \mu_1 - \mu_2 \geq \mu_0$ & $H_a: \mu_1 - \mu_2 < \mu_0$ (iii) $H_0: \mu_1 - \mu_2 \leq \mu_0$ & $H_a: \mu_1 - \mu_2 > \mu_0$
Step Two:	1. Quantitative *2. Random Sample* *3.* $n > 30$ OR the population is bell shaped
Step Three:	$z^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$
Step Four:	(i) $H_a: \mu_1 - \mu_2 \neq \mu_0$ → p-value = 2*P(Z<-|z*|) (ii) $H_a: \mu_1 - \mu_2 < \mu_0$ → p-value = P(Z<z*) (iii) $H_a: \mu_1 - \mu_2 > \mu_0$ → p-value = P(Z>z*) = 1-P(Z<z*)
Step Five:	If p-value$\leq (1 - confidene) = \alpha$ → Reject H_0 If p-value$> (1 - confidence) = \alpha$ → Fail to Reject H_0

Hypothesis Testing for μ unknown $\sigma_1 = \sigma_2$

Step One:	(i) $H_0: \mu_1 - \mu_2 = \mu_0$ & $H_a: \mu_1 - \mu_2 \neq \mu_0$ (ii) $H_0: \mu_1 - \mu_2 \geq \mu_0$ & $H_a: \mu_1 - \mu_2 < \mu_0$ (iii) $H_0: \mu_1 - \mu_2 \leq \mu_0$ & $H_a: \mu_1 - \mu_2 > \mu_0$
Step Two:	1. Quantitative *2. Random Sample* *3.* $n > 30$ OR the population is bell shaped
Step Three:	$t^* = \frac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where $s_p^2 = \frac{(\mathrm{n}_1 - 1)\mathrm{s}_1^2 + (\mathrm{n}_2 - 1)\mathrm{s}_2^2}{n_1 + n_2 - 2}$
Step Four:	(i) $H_a: \mu_1 - \mu_2 \neq \mu_0$→ p-value = 2*P(T<-|t*|) (ii) $H_a: \mu_1 - \mu_2 < \mu_0$→ p-value = P(T<t*) (iii) $H_a: \mu_1 - \mu_2 > \mu_0$→ p-value = P(T>t*) = 1-P(T<t*)
Step Five:	If p-value$\leq (1 - confidene) = \alpha$ → Reject H_0 If p-value$> (1 - confidence) = \alpha$ → Fail to Reject H_0

Hypothesis Testing for μ unknown $\sigma_1 \neq \sigma_2$

Step One:	(i) $H_0: \mu_1 - \mu_2 = \mu_0$ & $H_a: \mu_1 - \mu_2 \neq \mu_0$ (ii) $H_0: \mu_1 - \mu_2 \geq \mu_0$ & $H_a: \mu_1 - \mu_2 < \mu_0$ (iii) $H_0: \mu_1 - \mu_2 \leq \mu_0$ & $H_a: \mu_1 - \mu_2 > \mu_0$
Step Two:	1. Quantitative *2. Random Sample* *3.* $n > 30$ OR the population is bell shaped
Step Three:	$t^* = \dfrac{((\overline{x_1} - \overline{x_2}) - \mu_0)}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$
Step Four:	(i) $H_a: \mu_1 - \mu_2 \neq \mu_0$ → p-value = 2*P(T<-|t*|) (ii) $H_a: \mu_1 - \mu_2 < \mu_0$ → p-value = P(T<t*) (iii) $H_a: \mu_1 - \mu_2 > \mu_0$ → p-value = P(T>t*) = 1-P(T<t*)
Step Five:	If p-value$\leq (1 - confidene) = \alpha$ → Reject H_0 If p-value$> (1 - confidence) = \alpha$ → Fail to Reject H_0